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Inhomogeneous percolation problems and incipient infinite clusters

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Abstract. We consider inhomogeneous percolation models with density $p_c+f(x)$ and examine the forms of $f(x)$ which produce incipient structures. Taking $f(x) \approx |x|^{-\lambda}$ and assuming the existence of a correlation length exponent ν for the homogeneous percolation model, we prove that in $d=2$, the borderline value of λ is $\lambda_b = 1/\nu$. If $\lambda > 1/\nu$ then, with probability one, there is no infinite cluster, while if $\lambda < 1/\nu$ then, with positive probability, the origin is part of an infinite cluster. This result sheds some light on numerical and theoretical predictions of certain properties of incipient infinite clusters. Furthermore, for $d>2$, the models studied here suggest what sort of 'incipient objects' should be examined in random surface models.

1. Introduction

Much of the work on the critical behaviour of Bernoulli (independent) percolation is concerned with the properties of 'incipient infinite clusters'. A host of these objects have been discovered and phenotyped by a number of workers (see, e.g., Stanley (1977) Pike and Stanley (1981) and the general reviews of Stauffer (1979) and Essam (1980)); their scaling properties are considered crucial to the understanding of the critical regime in both percolation and related models (e.g. dilute ferromagnets and random resistor networks).

Despite the universal enthusiasm from the theoretical and numerical communities, there are very few rigorous results on the subject of incipient infinite clusters—in essence, the only exception being the work of Kesten (1986a)¶. In this paper, we examine an alternative proposal for the incipient infinite cluster, which can be analysed rigorously, and which has some features reminiscent of certain numerical and theoretical results.

Numerically, the question of 'what is an incipient infinite cluster?' is almost unnecessary; simulations of large samples performed at (or near) threshold produce

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¶ It was also suggested in Chayes *et al* (1986) and Chayes and Chayes (1985a) that the infinite-time invaded regions in the stochastic growth model known as invasion percolation may be suitable candidates for an 'incipient infinite cluster'. Such ideas have been entertained before (see, e.g., Wilkinson and Willemson 1983), but, as yet, no rigorous connection exists between invaded regions and incipient infinite clusters as defined in Kesten (1986a).

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enormous connected clusters and these objects are studied. Mathematically, the question is somewhat delicate. Indeed, it is expected on general grounds (and rigorously known in two dimensions (Russo 1981)) that, for short-range problems, the percolation transition is continuous. Whenever this is the case, with probability one, there can be no infinite object at the percolation threshold.

The proposal by Kesten (1986a) is to examine a sequence of finite volume conditional measures, constructed at the threshold density, which enjoy the property that the limiting measure contains an infinite connected object. For technical reasons—which are essentially the same ones that will plague us here—this programme has, so far, only achieved success in $d=2$.

An alternative idea is as follows. Consider a bond or site percolation problem on a regular lattice (henceforth taken to be $\mathbb{Z}^d$) with percolation threshold p_c. (Precise definitions will be provided in the next section.) As usual, we will take the bond (or site) 'occupation' probabilities to be statistically independent; however, we now allow these probabilities to be an inhomogeneous function of (lattice) position. In particular, we will consider densities of the form

$$\rho(x)=p_c+f(x) \tag{1}$$

where $f(x)\geq 0$ and $f(x)\to 0$ as $|x|\to\infty$. Moreover, we only consider functions f which do not affect the average density, i.e. we require that if Λ_L is an L^d-sized block centred at the origin, then

$$L^{-d}\sum_{x\in\Lambda_L} f(x)\to 0 \qquad \text{as} \qquad L\to\infty. \tag{2}$$

The goal in mind is to find a function f which tips the delicate balance—found in the uniform system—between the existence and absence of an infinite object. On the basis of 'folk wisdom', it is plausible that the interesting functions to consider are the power laws

$$f(x)\approx 1/|x|^{\lambda}. \tag{3}$$

Temporarily restricting attention to such functions, it is meaningful to ask whether there is a borderline value of λ, λ_b, such that when $\lambda>\lambda_b$, there is no infinite cluster (with probability one), while if $\lambda<\lambda_b$, the origin is connected to infinity with positive probability. If the answer is affirmative, one could study the properties of these infinite objects for powers smaller than (or perhaps including) λ_b.

Of course, we have fallen a little short of these goals; in particular, we are unable to treat the problem in dimensions larger than two. Furthermore, even in $d=2$, our proof of the existence of a critical power λ_b requires the existence of a correlation length exponent, ν, for the homogeneous problem. If it is the case that ν exists (in a sense which will be made precise in the next section), then it is somewhat surprising to learn that

$$\lambda_b=1/\nu. \tag{4}$$

Although our proof is restricted to $d=2$, it is probable that relation (4) holds more generally.

It is worth noting, as was pointed out to us by Stanley and Stauffer, that our result is reminiscent of the conclusions of Stanley (1977), Pike and Stanley (1981) and Coniglio (1981) that the dimension, d_r, of the 'singly connected' (or 'red') bonds of the incipient infinite cluster satisfies

$$d_r=1/\nu. \tag{5}$$

Although we do not yet have a compelling explanation as to why there should be a relation between these two results, it is not unlikely that this is the case, and that such a connection would provide some insight into the behaviour of these systems at threshold. We will therefore establish equation (4), modulo the proviso concerning the existence of ν.

We will devote the next section to some precise definitions (mainly to fix notation). In § 3, we will prove our principal result (theorem 2) and discuss some possible extensions to higher-dimensional problems. To simplify matters, we will confine our attention to the square bond lattice; our results, however, can be extended to several other two-dimensional models.

2. Preliminaries

Consider the site lattice $\mathbb{Z}^2$, and denote by $\mathbb{B}_2$ the set of all nearest-neighbour pairs (bonds) of $\mathbb{Z}^2$. Each bond $b \in \mathbb{B}_2$ will be labelled by the cartesian coordinate of its midpoint. Two bonds are said to be *connected* iff they have an endpoint in common.

Percolation on $\mathbb{B}_2$ is defined by declaring each bond $b \in \mathbb{B}_2$ to be occupied (or vacant) with probability p_b (resp $1-p_b$). These occupation events are generally taken to be independent and the p_b invariant under (lattice) translations. This provides us with a one-parameter family of problems depending on the value of $p_b \equiv p \in (0, 1)$.

Next, we consider the dual lattice $\mathbb{B}_2^*$, which is obtained by translating $\mathbb{B}_2$ half a unit in the x_1 and x_2 directions. Each $b^* \in \mathbb{B}_2^*$ is in one-to-one correspondence with the $b \in \mathbb{B}_2$ sharing its midpoint. Thus we may define the *dual model* by declaring b^* to be vacant when the corresponding b is occupied and vice versa.

Two sites, $x, y \in \mathbb{Z}^2$, are said to be *connected* iff there is a path of occupied bonds from x to y. The set of sites connected to a given site, x, will be called the *connected cluster* of x and denoted by $C(x)$. Observe that $C(x)$ is a random subset of $\mathbb{Z}^2$; its size (i.e. number of sites) will be denoted by $|C(x)|$.

Since the introduction of the percolation model (Broadbent and Hammersley 1957), it has been known that there is a critical value of p, p_c, satisfying $0<p_c<1$ such that

$$|C(0)|<\infty \qquad \text{with probability one} \tag{6a}$$

if $p<p_c$, while

$$|C(0)|=\infty \qquad \text{with non-zero probability} \tag{6b}$$

if $p>p_c$. This critical value of p is called the percolation threshold. In Kesten (1980), it was established that the dual model is also critical at (direct) bond density equal to p_c. (Since in this case the direct and dual models may be identified, this implies $p_c=\frac{1}{2}$.)

For values of $p<p_c$, it is of interest to consider the so-called connectivity:

$$\tau_{0x} = \text{prob}(x \in C(0)) \tag{7}$$

while for $p>p_c$, one either looks at the (dual) connectivity between points on the dual lattice:

$$\tau^*_{0^*x^*} = \text{prob}(x^* \text{ connected to } 0^* \text{ by occupied dual bonds}) \tag{8}$$

or the truncated connectivity:

$$\tau'_{0x} = \text{prob}(x \in C(0) \text{ and } |C(0)|<\infty). \tag{9}$$

It is straightforward to establish (see e.g. Kesten 1982, Chayes and Chayes 1985a) that the limits

$$\xi^{-1}(p) \equiv -\lim_{|x|\uparrow\infty} (|x|^{-1} \log \tau_{0x}) \tag{10a}$$

and

$$[\xi^*(p)]^{-1} \equiv -\lim_{|x^*|\uparrow\infty} [|x|^{-1} \log \tau^*_{0^*x^*}] \tag{10b}$$

exist; $\xi(p)$ is called the *correlation length* and satisfies, for finite x, the inequality $\tau_{0x}(p) \leq \exp(-x/\xi(p))$. A similar estimate is available for the quantity $\xi^*(p)$. According to Grimmett (1981, 1983), $\xi^{-1}(p)$ is a continuous function which, unless zero, is strictly decreasing. Corresponding statements are also valid for $(\xi^*(p))^{-1}$. As for the *truncated* connectivities, it is also possible to show (Chayes and Chayes 1985b) that

$$(\xi'(p))^{-1} \equiv -\lim_{|x|\uparrow\infty} (|x|^{-1} \log \tau'_{0x}) \tag{10c}$$

exists, and that in fact $\xi'(p) = \frac{1}{2}\xi^*(p)$. Obviously, in the self-dual model $\xi^*(p) = \xi(1-p)$, so that in particular $\xi'(p) = \frac{1}{2}\xi(1-p)$.

A long time ago, it was shown (Hammersley 1957; see also Aizenman and Newman 1984) that

$$\xi(p) < \infty \Leftrightarrow E[|C(0)|] < \infty. \tag{11}$$

The essence of the result of Kesten (1980) is that the correlation length (and hence $E[|C(0)|]$) diverges exactly at p_c. Thus

$$\xi(p) < \infty \Leftrightarrow p < p_c. \tag{12}$$

This has been recently extended to general dimension (Aizenman and Barsky 1986).

It is natural to suspect that the divergence of the correlation length is characterised by a power law: $\xi(p) \approx |p-p_c|^{-\nu}$. Such a result has not yet been established with complete rigour for any dimension. (Some progress on this question can be found in Kesten (1981), Durrett and Nguyen (1985) and Nguyen (1985); see also Chayes *et al* (1985a).) It is perhaps not too much to hope that, in general, the limit

$$\nu \equiv \lim_{p\uparrow p_c} |\log \xi(p)/\log(p_c - p)| \tag{13}$$

exists.

The correlation length as defined above is somewhat abstract. A natural (physical) notion of the correlation length is as follows: assume that $\xi(p) < \infty$, and consider the $L \times L$ box Λ_L. Let $R_L(p)$ be the probability that there is a crossing from the left face of Λ_L to the right (see figure 1). It is straightforward to establish that if $L \gg \xi(p)$, then

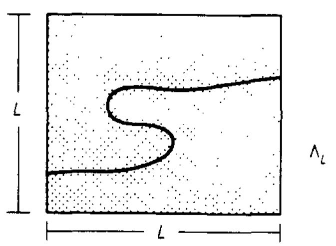

Figure 1. A left-right crossing.

$R_L(p)$ is small; indeed, $R_L(p) \sim \exp(-L/\xi(p))$. On the other hand, at very small distances, the discreteness of the lattice dictates that $R_L(p)$ will be of order one, essentially independent of p (i.e. as we move through the critical regime). This implies that there must be a *crossover* scale where $R_L(p)$ first becomes 'small' by some fixed criterion. It is an overwhelming temptation to identify this scale with the correlation length as defined by equation (10). That this is indeed possible has been demonstrated by Chayes *et al* (1985a) (see also Chayes and Chayes 1985a).

Proposition. 1. There is a constant $0 < c < 1$ such that if $L(p)$ is the smallest scale where $R_L(p) \leq c$, then, up to logs, $L(p)$ is the correlation length, in the sense that

$$a_1 L(p)/\log L(p) \leq \xi(p) \leq a_2 L(p)$$

for some positive finite constants a_1 and a_2.

Proof. See the above references.

In particular, proposition 1 tells us that if $\xi(p)$ diverges with critical exponent ν in the sense of equation (13), so does $L(p)$. It is worth remarking that proposition 1 goes through more or less intact in higher dimensions. We now possess all the machinery necessary to prove our principal result.

3. Main results

Let us first outline the strategy of the proof. In the preceding section, we learned that, at the scale of the correlation length, crossing probabilities of rectangles are small but not unreasonably so. Similarly, for $p \geq p_c$, at the scale of the truncated or dual correlation length, it is not unlikely to observe crossings of rectangles by dual bonds. This (at least in $d = 2$) is what prevents the existence of an infinite cluster at $p = p_c$.

In order to illustrate the above reasoning, let us divide the two-dimensional universe into disjoint annuli, each one (say) three times the size of the one before (see figure 2). Let $A_L^*(p)$ be the probability, at bond density p, that there is a circuit of dual bonds in the annulus $\Lambda_{3L} \backslash \Lambda_L$. It is known (Russo 1978, Seymour and Welsh 1978)

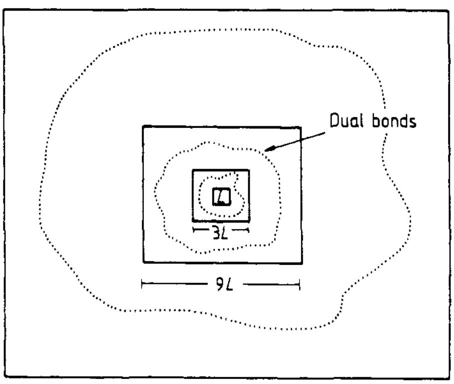

Figure 2. Circuits in annuli.

that $A_L^*(p)$ can be bounded below by a function of the dual crossing probability, $R_L^*(p) = 1 - R_L(p)$; explicitly,

$$A_L^*(p) \geq [(\{[1-(1-R_L^*(p))^{1/2}]^3\}^2 R_L^*(p))^2 R_L^*(p)]^4. \tag{14}$$

Thus, if $R_L^*(p)$ is not terribly small, neither is $A_L^*(p)$. If $p \geq p_c$, one can therefore bank on observing on the order of $\log \xi^*(p)$ (independent) annuli surrounding the origin which contain a cutting circuit of dual bonds. Beyond this distance is where one runs into the infinite cluster; inside it, the infinite cluster is hard-pressed—indeed, the probability of observing an infinite cluster *at the origin* is bounded above and below by inverse powers of $\xi^*(p)$. Hence, at $p = p_c$, the divergence of the correlation length prevents the existence of an infinite cluster.

Now, as a function of $\not{p} \equiv p - p_c$, $\xi'(\not{p})$ relates density to a distance scale. Since $\xi'(\not{p})$ is monotone, we may invert the function (i.e. define, unambiguously, $\not{p}(\xi')$.) Thus, if we consider the inhomogeneous density

$$\rho(x) = \rho_c + \not{p}(|x|) \tag{15}$$

we find that each distance scale feels that it is right at the scale of 'the correlation length' (perhaps to within logarithmic factors). Any further tinkering (as we do below) has the effect of making each scale act as though it is well within or well outside the correlation length.

Theorem 2. For all $\varepsilon > 0$, if $\rho(x) = p_c + \not{p}(|x|^{1+\varepsilon})$, then with probability one, the origin is in a finite cluster, whereas if $\rho(x) = p_c + \not{p}(|x|^{1-\varepsilon})$, then with non-zero probability, the origin is connected to infinity. In the latter case, there is a unique infinite cluster with probability one.

Proof. Take $\varepsilon \in \mathbb{R}^+$. Let us consider the inhomogeneous density $\rho(x) = \rho_c + \not{p}(|x|^{1+\varepsilon})$. Observe that in the annular regions $\Lambda_{3L} \backslash \Lambda_L$, the density is no larger than $p_c + \not{p}(|L|^{1+\varepsilon})$. At this density, in a uniform system, the probability of observing that a square of size L is crossed by dual bonds is bounded below independent of L—were this not the case, then by the considerations of proposition 1, something smaller than L would have been the correlation length, not $L^{1+\varepsilon}$! From this, and the bound of equation (14), we see that the probabilities $A_L^*(\rho)$ are bounded below uniformly in L. (This, as well as many other results cited, requires the use of the Harris-FKG inequality (Harris 1960, Fortuin *et al* 1971). For our purposes, this means that the probabilities $A_L^*(\rho)$, which involve the cooperative activity of *dual bonds*, is at least as large as $A_L^*(\bar{\rho})$, where $1-\bar{\rho}$ is the minimum dual bond probability found anywhere in the annulus.) Hence, if we divide $\mathbb{Z}^2$ into disjoint scales as depicted in figure 2, with probability one, infinitely many of the annuli contain circuits of dual bonds. This disconnects the origin from infinity with probability one.

Next, consider the inhomogeneous density $\rho(x) = p_c + \not{p}(|x|^{1-\varepsilon})$. In this case, we will exploit the excess density to construct, by hand, an infinite cluster. To this end, consider a sequence of lengths $L_0, L_1, \ldots$ satisfying $L_{n+1} = 3L_n$. Let V_n and H_n be the vertical and horizontal regions given by

$$V_n = \left\{ x \in \mathbb{Z}^2 \middle| \sum_{i=0}^{n-1} L_i \leq x_1 \leq \sum_{i=0}^{n} L_i;\ |x_2| \leq \tfrac{3}{2} L_n \right\} \tag{16}$$

and

$$H_n = \left\{ x \in \mathbb{Z}^2 \middle| \sum_{i=0}^{n-1} L_i \leq x_1 \leq \sum_{i=0}^{n+1} L_i;\ |x_2| \leq \tfrac{1}{2} L_n \right\} \tag{17}$$

and define $T_n = V_n \cup H_n$. (The reader is urged to consult figure 3.) Observe that the T_n have well tailored overlaps (as depicted in figure 4), and extend to infinity. Now, the density inside T_n is at least as large as $p_c + \not{h}(|7L_n|^{1-\varepsilon})$. Hence, by definition, the dual correlation length is of order $L_n^{1-\varepsilon}$†.

Consider the event $\mathscr{V}_n$ that there is a top-bottom crossing of V_n by direct bonds. The only thing that prevents $\mathscr{V}_n$ from occurring is a left-right crossing of V_n by dual bonds. By considering dual crossings between all possible pairs of points on the left and right faces of V_n, it is not hard to see that the probability of the latter is bounded above by $c_1 L_n^2 \exp(-c_2 L_n / L_n^{1-\varepsilon})$, where c_1 and c_2 are positive finite constants independent of n. Hence,

$$\text{prob}_{p(x)}(\mathscr{V}_n) \geq 1 - c_1 L_n^2 \exp(-c_2 L_n^{\varepsilon}). \tag{18}$$

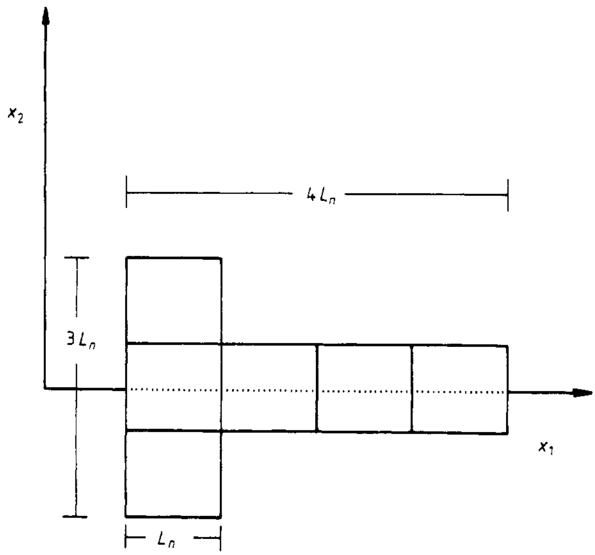

Figure 3. The region T_n.

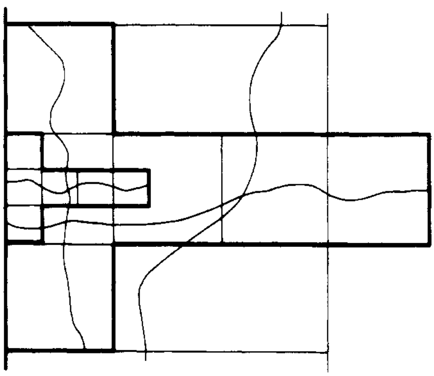

Figure 4. The T construction.

† Again, we can use the Harris-FKG inequality to bound the correlation length inside T_n (above and below) by the value it would have in the homogeneous systems with the densities of the best and worst case scenarios.

An identical argument shows that the event $\mathscr{H}_n$ that there is a left–right crossing of H_n by occupied bonds has probability bounded below by an estimate qualitatively similar to the one in equation (18). Thus, if we consider the event $\mathscr{T}_n = \mathscr{V}_n \cap \mathscr{H}_n$ that 'the T is crossed', we can find positive finite constants d_1 and d_2 such that, uniformly in n,

$$\mathrm{prob}_{\rho(x)}(\mathscr{T}_n) \geq 1 - d_1 L_n^2 \exp(-d_2 L_n^{\varepsilon}). \tag{19}$$

From (19), it is not hard to show that, with probability one, all but a finite number of the T are crossed. (Here we have used the Borel–Cantelli lemma.) This implies, with probability of order one, the presence of an infinite cluster a finite distance from the origin—local fluctuations will now serve us to get the *origin* connected to infinity with non-zero probability.

Furthermore, if we do the T construction simultaneously along all four coordinate directions, we see that (with probability one) the origin is surrounded infinitely often by occupied circuits. This establishes the uniqueness of the infinite cluster.

Corollary. If, for the $\mathbb{B}_2$ homogeneous percolation systems the correlation length exponent ν exists in the sense of equation (13), then for inhomogeneous densities of the form

$$\rho(x) = p_c + f(x)$$

with $f(x) \sim 1/|x|^{\lambda}$, the borderline value of λ for the existence of an infinite cluster is $\lambda_b = 1/\nu$.

4. Concluding remarks

(i) The reader will observe that we have attacked the *dual* model more vehemently than the direct model. A more traditional statement of the corollary to theorem 2 should therefore be $\lambda_b = 1/\nu'$ for $d = 2$. In self-dual models, it goes without saying that $\nu' = \nu$ (in any sense in which either should exist). For all other two-dimensional lattices where we can prove the above theorem, the relationship $\nu' = \nu$ has recently been established by Kesten (1986b).

(ii) As the above remark indicates, part of what enabled us to prove theorem 2 is that, in two dimensions, the dual of bond percolation is bond percolation. (This is inarguable for self-dual models, and morally true in other two-dimensional systems.) A manifestly different situation is encountered in $d > 2$. For example, in $d = 3$, the model dual to bond percolation on $\mathbb{Z}^3$ is that generated by independently occupied *plaquettes*. Furthermore, the relevant dual transition is not plaquette percolation. Indeed, as was shown by Aizenman *et al* (1983), the relevant transition concerns the formation of infinite sheets of plaquettes.

In spite of the utility of the incipient infinite cluster in bond models, it seems that no one has yet addressed the question of the relevant incipient object for random surface models. Indeed, it is not even clear how one should define such an object. Since, at the threshold of the surface-dominated regime, we are well beyond the point of plaquette percolation, there will already be a (very) dense infinite cluster of plaquettes. This must somehow be pared down.

In bond percolation problems, one can always look at the 'backbone' (incipient or otherwise) of infinite structures. After a moment's thought, it is realised that a backbone is an infinite object that has no 'boundary'—in the sense that none of its bonds have only a single exposed vertex. Is there an 'incipient infinite surface' which is without 'boundary' (i.e. without any exposed $d-2$ cells) and surrounds any finite region at the threshold of the surface-dominated regime? If so, what are the properties of this surface? It may be the case that something satisfying the above criterion *is already present* (i.e. with positive density) at threshold due to a 'polymer-like' condensation of surface filaments (tubes) at lower plaquette densities. If so, such an effect would have to be disentangled.

The methods of this paper can easily be applied to an 'incipient backbone'; such an object could have been constructed with inhomogeneous density $p_c + \not{p}(|x|^{1-\varepsilon})$ simply by running the 'T construction' simultaneously to the left and right. A more interesting issue is the extension of our results to the surface models in higher dimensions. For example, if we consider, in $d=3$, inhomogeneous plaquette densities of the form $q(x) = (1-p_c) + \not{p}(|x|^{1-\varepsilon})$, one can construct, by extensions of theorem 2, an enormous 'incipient surface' with no boundary. In such cases, one would find an analogue of theorem 2, i.e. the borderline power law would again be $1/\nu$. How much of this surface (if any) is the relevant object for understanding the critical point behaviour of Bernoulli systems from the point of view of the surfaces is an open question. In any case, in accordance with the traditional ideas of Widom (1965), the above result indicates that a correlation length exponent for a given problem may have 'something to do' with the associated random surface problem.

Acknowledgments

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